

THE MODULI SPACE OF HESSIAN QUARTIC SURFACES AND AUTOMORPHIC FORMS

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ABSTRACT. We shall show the existence of 15 automorphic forms of weight 8 on the moduli space of marked Hessian quartic surfaces of cubic surfaces. These 15 automorphic forms correspond to $(\lambda_i - \lambda_j)(\lambda_k - \lambda_l)$ where $\lambda_i, \lambda_j, \lambda_k, \lambda_l$ ($1 \leq i, j, k, l \leq 5$) are the coefficients of the Sylvester form of a general cubic surface and all i, j, k, l are distinct.

1. INTRODUCTION

The purpose of this note is to give an example of automorphic forms on the moduli space of Hessian quartic surfaces which can be interpreted in terms of invariants of cubic surfaces. Let S be a smooth cubic surface defined by a homogeneous polynomial $F(z_0, z_1, z_2, z_3)$ of degree 3. Then the hessian of F , if it is not identically zero, defines a quartic surface H called the *Hessian quartic surface* of S . To study Hessian quartic surfaces, it is convenient to use the Sylvester form of S . It is classically known (cf. Segre [23], Chap. IV) that a general cubic surface can be written by the Sylvester form:

$$\lambda_1 x_1^3 + \cdots + \lambda_5 x_5^3 = 0, \quad x_1 + \cdots + x_5 = 0.$$

A general cubic surface defined by the Sylvester form is uniquely determined by

$$\lambda = (\lambda_1 : \cdots : \lambda_5) \in \mathbf{P}^4$$

up to permutation of λ_i . By using the theory of periods of $K3$ surfaces (Piatetskii-Shapiro, Shafarevich [22]), one can describe the moduli space of Hessian quartic surfaces as an arithmetic quotient of the 4-dimensional bounded symmetric domain of type IV. Koike [17] gave an \mathfrak{S}_5 -equivariant birational map from \mathbf{P}^4 to the moduli space of marked Hessian quartic surfaces. On the other hand, Dardanelli and van Geemen [12] studied the transcendental lattices of the Hessian quartic surfaces of cubic surfaces with a node, with an Eckardt point or without a Sylvester form. In the moduli space of marked Hessian quartic surfaces, the Heegner divisor corresponding to cubic surfaces with an Eckardt point consists of 10 irreducible components. A cubic surface defined by a Sylvester form has an Eckardt point iff $\lambda_i = \lambda_j$ for some $i \neq j$. Thus 10 components defined by $\lambda_i = \lambda_j$ in \mathbf{P}^4 bijectively correspond to 10 components of the above Heegner divisor in the moduli space of marked Hessian quartic surfaces (Lemma 4.2). The minimal model of a general Hessian quartic surface has a canonical fixed point free involution (for example, see Dolgachev, Keum [14]), and hence it is the covering $K3$ surface of an Enriques surface. Thus the moduli space of cubic surfaces is birational to the moduli space of Enriques surfaces whose covering $K3$ surfaces are Hessian quartic surfaces. In the paper [19], the author gave $3^3 \cdot 5 \cdot 17 \cdot 31$ holomorphic automorphic forms F_V of weight 4 with known zeros on the moduli space of marked Enriques surfaces by using Borchers theory of automorphic forms [6].

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In this note, by first dividing F_V by a product of suitable linear forms and restricting it to the locus of marked Enriques surfaces corresponding to marked Hessian quartic surfaces, we have 15 automorphic forms of weight 8. We shall show that under the birational map 15 automorphic forms correspond to $(\lambda_i - \lambda_j)(\lambda_k - \lambda_l)$ where all i, j, k, l are distinct (Theorem 6.2).

The Hessian quartic surface is an example of the Cayley symmetroid given by the zero of a 4×4 symmetrical determinant whose entries are the linear form (Dardanelli, van Geemen [12], §1.6). A general Cayley symmetroid is the covering $K3$ surface of a nodal Enriques surface, namely an Enriques surface containing a smooth rational curve (Cossec [11]). The period domain of nodal Enriques surfaces is the 9-dimensional bounded symmetric domain \mathcal{D} of type IV which is naturally embedded in $\mathcal{D}(L_-)$. By the similar way as in the case of Hessian quartic surfaces, we have $3^3 \cdot 5 \cdot 17$ holomorphic automorphic forms on \mathcal{D} . It would be interesting to study a relation between these automorphic forms and the geometry of Cayley symmetroids given in Coble [9], Chapters V, VI.

Finally we mention the related works. In the paper [1], Allcock, Carlson, Toledo showed that the moduli space of cubic surfaces can be described as an arithmetic quotient of the 4-dimensional complex ball by considering the period of the intermediate Jacobian of the triple cover of \mathbf{P}^3 branched along a cubic surface. Later Dolgachev, van Geemen and the author [13] gave the same description of the moduli space of cubic surfaces by using the theory of periods of $K3$ surfaces. By using this description of the moduli space and Borchers' theory of automorphic forms [6], Allcock, Freitag [2] studied the moduli space of marked cubic surfaces. They constructed automorphic forms corresponding to Cayley's cross ratios for cubic surfaces.

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2. PRELIMINARIES

A *lattice* (L, \langle, \rangle) is a pair of a free \mathbf{Z} -module L of rank r and a non-degenerate symmetric integral bilinear form $\langle, \rangle : L \times L \rightarrow \mathbf{Z}$. For simplicity we omit \langle, \rangle if there are no confusions. For $x \in L \otimes \mathbf{Q}$, we call $x^2 = \langle x, x \rangle$ the *norm* of x . For a lattice (L, \langle, \rangle) and an integer m , we denote by $L(m)$ the lattice $(L, m\langle, \rangle)$. We denote by U the even unimodular lattice of signature $(1, 1)$, and by A_m , D_n or E_k the even *negative* definite lattice defined by the Cartan matrix of type A_m , D_n or E_k respectively. For an integer m , we denote by $\langle m \rangle$ the lattice of rank 1 generated by a vector with norm m . We denote by $L \oplus M$ the orthogonal direct sum of lattices L and M .

Let L be an even lattice and let $L^* = \text{Hom}(L, \mathbf{Z})$. We denote by A_L the quotient L^*/L and define a map

$$q_L : A_L \rightarrow \mathbf{Q}/2\mathbf{Z}$$

by $q_L(x + L) = \langle x, x \rangle \bmod 2\mathbf{Z}$. We call q_L the *discriminant quadratic form* of L . We denote by u or v the discriminant quadratic form of $U(2)$ or D_4 respectively.

Let $O(L)$ be the orthogonal group of L , that is, the group of isomorphisms of L preserving the bilinear form. Similarly $O(q_L)$ denotes the group of isomorphisms of A_L preserving q_L . There is a natural map

$$O(L) \rightarrow O(q_L)$$

whose kernel is denoted by $\tilde{O}(L)$.

3. THE HESSIANS OF CUBIC SURFACES AND ENRIQUES SURFACES

Let S be a smooth cubic surface defined by a homogeneous polynomial $F(z_0, z_1, z_2, z_3)$ of degree 3. Then the hessian of F , if it is not identically zero, defines a quartic surface H called the Hessian quartic surface of S . It is classically known that a general cubic surfaces S can be written by the *Sylvester form*

$$(3.1) \quad \lambda_1 x_1^3 + \cdots + \lambda_5 x_5^3 = 0, \quad x_1 + \cdots + x_5 = 0$$

where x_1, \dots, x_5 are linear forms in z_0, z_1, z_2, z_3 each four of them are linearly independent and $\lambda_i \in \mathbf{C}^*$. The forms x_1, \dots, x_5 are uniquely determined by F up to permutation and multiplication by a common non-zero scalar, and $\lambda_1, \dots, \lambda_5$ are uniquely determine by F and x_i . Thus a general cubic surface defined by the Sylvester form is now determined by

$$\lambda = (\lambda_1 : \cdots : \lambda_5) \in \mathbf{P}^4$$

up to permutations of λ_i (Segre [23], Chap. IV).

For a cubic surface defined by the Sylvester form, the corresponding Hessian quartic surface H is given by

$$(3.2) \quad \frac{1}{\lambda_1 x_1} + \cdots + \frac{1}{\lambda_5 x_5} = 0, \quad x_1 + \cdots + x_5 = 0.$$

The Hessian quartic surface H has 10 nodes p_{ijk} defined by $x_i = x_j = x_k = 0$, and contains 10 lines l_{mn} defined by $x_m = x_n = 0$. It is known (Segre [23], Chap. IV) that H has no other singular points if and only if

$$(3.3) \quad \Delta_{\text{sing}}(\lambda) = \sum_{i=1}^5 \frac{1}{\pm \sqrt{\lambda_i}} \neq 0.$$

We denote by X the minimal resolution of H which is a $K3$ surface with 20 smooth rational curves, that is, exceptional curves E_{ijk} over 10 nodes p_{ijk} and strict transforms L_{mn} of 10 lines l_{mn} . The curve E_{ijk} meets exactly three curves L_{ij} , L_{ik} and L_{jk} , and conversely L_{ij} meets exactly three curves E_{ijk} ($k \neq i, j$). Thus we have two sets $\{E_{ijk}\}, \{L_{mn}\}$ of smooth rational curves on X each of which consists of 10 disjoint curves, and each curve in one set meets exactly three curves in the other set.

The birational involution defined by

$$(3.4) \quad (x_1 : \cdots : x_5) \rightarrow \left(\frac{1}{\lambda_1 x_1} : \cdots : \frac{1}{\lambda_5 x_5} \right)$$

induces a fixed point free involution σ of X , and hence the quotient $Y = X/\langle \sigma \rangle$ is an Enriques surface (Dolgachev, Keum [14]). The involution σ switches nodal curves E_{ijk} and L_{mn} where $\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$. We denote by \bar{L}_{ij} the image of L_{ij} or E_{kmn} on Y . The curve \bar{L}_{ij} meets exactly three curves \bar{L}_{km} , \bar{L}_{kn} and \bar{L}_{mn} . We can easily see that the dual graph of ten nodal curves $\{\bar{L}_{ij}\}$ is isomorphic to the Petersen graph whose automorphism group is the symmetry group \mathfrak{S}_5 of degree 5.

Denote by $\pi : X \rightarrow Y$ the natural projection and let L be the lattice $H^2(X, \mathbf{Z})$ which is the even unimodular lattice of signature $(3, 19)$. Define

$$(3.5) \quad L_{\pm} = \{x \in L \mid \sigma^*(x) = \pm x\}.$$

It is known that $L_+ \cong \pi^*(H^2(Y, \mathbf{Z})) \cong U(2) \oplus E_8(2)$ and $L_- \cong U \oplus U(2) \oplus E_8(2)$ (Barth, Peters [4]).

The 20 curves $\{E_{ijk}, L_{mn}\}$ generate a sublattice N of signature $(1, 15)$ in the Picard lattice S_X of X . Let M be the orthogonal complement of N in L . It is known that M is isomorphic to $U \oplus U(2) \oplus A_2(2)$ (Dolgachev, Keum [14]). Let R be the orthogonal complement of M in L_- . Obviously R is a negative definite lattice of rank 6.

3.1. Lemma. *R is isomorphic to $E_6(2)$.*

Proof. For any smooth rational curve C on Y , $\pi^*(C)$ is the disjoint union of two smooth rational curves. The difference of these two curves is a vector of norm (-4) contained in R . For example, $E_{123} - L_{45}$, $E_{145} - L_{23}$, $E_{235} - L_{14}$, $E_{345} - L_{12}$, $E_{125} - L_{34}$, and $E_{245} - L_{13}$ generate a lattice isomorphic to $E_6(2)$ in R . By comparing $A_{L_-} = (\mathbf{Z}/2\mathbf{Z})^{10}$ and $A_{E_6(2) \oplus M} = (\mathbf{Z}/2\mathbf{Z})^{10} \oplus (\mathbf{Z}/3\mathbf{Z})^2$, we can conclude that $E_6(2)$ is the orthogonal complement of M in L_- . \square

3.2. Remark. *Nikulin [21] introduced the notion of root invariant (R', K) of each Enriques surface consisting of a root lattice R' and a finite subgroup K of $R'/2R'$. The $R'(2)$ is generated by the differences $C - \pi^*(C)$ of all smooth rational curves C on the Enriques surface. In case that the covering K3 surface of Y is a Hessian quartic surface X , the generic Y has the root invariant $(E_6, \{0\})$.*

Let S_X be the Picard lattice of X . The orthogonal complement of S_X in $H^2(X, \mathbf{Z})$, denoted by T_X , is called the transcendental lattice of X . For a generic cubic surface S , N (resp. M) coincides with S_X (resp. T_X) (Dolgachev, Keum [14]).

Recall that a smooth cubic surface S has 45 tritangent planes each of which consists of three lines. If three coplanar lines meet at one point, the intersection point is called an *Eckardt point*. A smooth cubic surface S given by the Sylvester form (3.1) has an Eckardt point if and only if $\lambda_i = \lambda_j$ (Segre [23], Chap. IV). If S has an Eckardt point, for example $\lambda_i = \lambda_j$, the tritangent plane is given by $x_i + x_j = 0$ and the Eckardt point on S is p_{kmn} where $\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$. In this case, the plane section defined by $x_i + x_j = 0$ on the Hessian quartic surface consists of $2l_{ij}$ and two lines through the node p_{kmn} (see [12], §2.1, 2.2). The strict transforms of the two lines to X are two disjoint smooth rational curves N_{ij}^+, N_{ij}^- . The involution σ switches N_{ij}^+ and N_{ij}^- . The curves N_{ij}^\pm meet L_{ij} and E_{kmn} with multiplicity 1 and disjoint with other 18 curves. Thus we have the following Lemma.

3.3. Lemma. *If a smooth cubic surface has an Eckardt point corresponding to $\lambda_i = \lambda_j$, then X contains new two smooth rational curves N_{ij}^+ and N_{ij}^- . The class of $N_{ij}^+ - N_{ij}^-$ is a (-4) -vector contained in M .*

Proof. Note that $N_{ij}^+ - N_{ij}^-$ is perpendicular to 20 smooth rational curves $\{L_{ij}, E_{kmn}\}$. Since 20 curves $\{L_{ij}, E_{kmn}\}$ generate N , we have $N_{ij}^+ - N_{ij}^- \in N^\perp = M$. \square

3.4. Remark. *If all $\lambda_i = 1$, the cubic surface is called the Clebsch diagonal cubic surface which has 10 Eckardt points (see [12], Lemma 2.2). The corresponding Enriques surface contains 20 smooth rational curves and the symmetry group \mathfrak{S}_5 of degree 5 acts on the Enriques surface as automorphisms. This Enriques surface is one of Enriques surfaces with a finite group of automorphisms classified in [18] (see [18], Example VI, [12], §2.3).*

4. DISCRIMINANT QUADRATIC FORM

First recall that N and M are primitive sublattices of the even unimodular lattice $L = H^2(X, \mathbf{Z})$ with $M = N^\perp$. It follows from Nikulin [20], Corollary 1.6.2 that $q_M \cong -q_N$. By an elementary calculation, $A_M \cong (\mathbf{Z}/2\mathbf{Z})^4 \oplus \mathbf{Z}/3\mathbf{Z}$ and the restriction $(q_M)_2$ of q_M to the 2-Sylow subgroup of A_M is isomorphic to $u \oplus v$. We can consider $(q_M)_2$ a 4-dimensional quadratic form over \mathbf{F}_2 . It is well known that the group of automorphisms of the quadratic form $(q_M)_2$ is isomorphic to the symmetry group \mathfrak{S}_5 of degree 5 ([10], page 2), and hence $O(q_M)$ is isomorphic to $\mathfrak{S}_5 \times \mathbf{Z}/2\mathbf{Z}$ where $\mathbf{Z}/2\mathbf{Z}$ is the involution of $\mathbf{Z}/3\mathbf{Z}$. It is easily seen that $(q_M)_2$ contains $2(2^2 - 1)$ isotropic vectors and $2(2^2 + 1)$ non-isotropic vectors. For $a \in A_M$ we denote by $|a|$ the order of a . We can easily see the following lemma.

4.1. Lemma. *The number of vectors a in A_M with*

$(|a|, q_M(a)) = (0, 0), (2, 0), (2, 1), (3, -4/3), (6, -1/3)$ or $(6, -4/3)$ is 1, 5, 10, 2, 20 or 10, respectively.

It follows from Lemma 4.1 that $A_N (\cong A_M)$ contains 10 vectors with norm 1. In the following we shall study a geometric meaning of these 10 vectors. Recall that the Enriques surface Y contains 10 smooth rational curves \bar{L}_{ij} . If we fix \bar{L}_{ij} one of them, then there are 6 smooth rational curves perpendicular to \bar{L}_{ij} which form a singular fiber of type I_6 of an elliptic fibration. We denote this elliptic fibration by $|\bar{F}_{ij}|$. For example, if we take \bar{L}_{12} , then the class

$$\bar{F}_{12} = \bar{L}_{13} + \bar{L}_{24} + \bar{L}_{15} + \bar{L}_{23} + \bar{L}_{14} + \bar{L}_{25} = \bar{E}_{245} + \bar{E}_{135} + \bar{E}_{234} + \bar{E}_{145} + \bar{E}_{235} + \bar{E}_{134}$$

defines an elliptic fibration on Y , and \bar{L}_{12} is a component of another singular fiber of the fibration. Then $\pi^*(\bar{F}_{ij}) = 2F_{ij}$ and the class

$$F_{ij} = L_{13} + E_{135} + L_{15} + E_{145} + L_{14} + E_{134} = E_{245} + L_{24} + E_{234} + L_{23} + E_{235} + L_{25}$$

defines an elliptic fibration on X . We can easily see that

$$\alpha_{ij} = \frac{1}{2}(F_{ij} - L_{ij} - E_{kmn}), \quad \{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$$

has an integral intersection number with any curve from 20 curves $\{L_{ij}, E_{kmn}\}$. Since 20 curves $\{L_{ij}, E_{kmn}\}$ generate N , α_{ij} is contained in N^* . Obviously $q_N(\alpha_{ij}) = 1$.

In the Sylvester form (3.1), if $\lambda_i = \lambda_j$, then S has an Eckardt point p_{kmn} . We have two new smooth rational curves N_{ij}^+ , N_{ij}^- meeting L_{ij} and E_{kmn} (Lemma 3.3). Recall that the involution σ switches N_{ij}^+ and N_{ij}^- . Let \bar{N}_{ij} be the image of N_{ij}^+ on Y . Then $\bar{N}_{ij} + \bar{L}_{ij}$ is a singular fiber of $|\bar{F}_{ij}|$ of type I_2 . Note that this is a multiple fiber because $N_{ij}^+ + N_{ij}^- + L_{ij} + E_{kmn}$ is a singular fiber of type I_4 of the elliptic fibration $|F_{ij}|$. It follows that

$$\alpha_{ij} = \frac{1}{2}(N_{ij}^+ + N_{ij}^-).$$

The difference

$$\beta_{ij} = \frac{1}{2}(N_{ij}^+ - N_{ij}^-)$$

defines a vector in M^* with $q_M(\beta_{ij}) = 1$ (Lemma 3.3). The condition

$$\alpha_{ij} + \beta_{ij} \in L = H^2(X, \mathbf{Z})$$

gives a bijective correspondence between the set of vectors with norm 1 in A_M and that in A_N .

Moreover if $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, but other coefficients are different, then S has exactly two Eckardt points. In this case, β_{12} is perpendicular to β_{34} (see Dardanelli, van Geemen [12], Lemma 2.2, the case $k = 2$). Thus we have the following Lemma.

4.2. Lemma. *There is a bijective correspondence between the 10 conditions $\lambda_i = \lambda_j$ between $\{\lambda_i\}$ and the set of vectors in A_M with norm 1. Moreover if S has exactly two Eckardt points corresponding to $\lambda_i = \lambda_j$ and $\lambda_k = \lambda_m$ where all i, j, k, m are distinct, then β_{ij} is perpendicular to β_{km} .*

5. PERIODS AND HEEGNER DIVISORS

In this section, we recall the period domain and Heegner divisors for Enriques surfaces and Hessian quartic surfaces. First we consider the case of Enriques surfaces. Define

$$(5.1) \quad \mathcal{D}(L_-) = \{[\omega] \in \mathbf{P}(L_- \otimes \mathbf{C}) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}$$

which is a disjoint union of two copies of the 10-dimensional bounded symmetric domain of type IV.

The discriminant quadratic form (A_{L_-}, q_{L_-}) is the orthogonal direct sum of five copies of u . We consider the orthogonal group $O(L_-)$ of L_- and denote by $\tilde{O}(L_-)$ the kernel of the map

$$O(L_-) \rightarrow O(q_{L_-}).$$

Then $O(L_-)/\tilde{O}(L_-) \cong O(q_{L_-}) \cong O^+(10, \mathbf{F}_2)$ (Barth, Peters [4]).

For a vector $r \in L_-$ with a negative norm, we put

$$r^\perp = \{[\omega] \in \mathcal{D}(L_-) \mid \langle \omega, r \rangle = 0\}.$$

Let $a \in A_{L_-}$ be a non-isotropic vector, that is, $q_{L_-}(a) = 1$. We define Heegner divisors $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}_a$ by

$$\tilde{\mathcal{H}} = \sum_r r^\perp, \quad \tilde{\mathcal{H}}_a = \sum_t t^\perp$$

where r moves over the set of all (-2) -vectors in L_- and t moves over the set of all (-4) -vectors in L_- satisfying $\frac{t}{2} \bmod L_- = a$. It is known that $\mathcal{D}(L_-) \setminus \tilde{\mathcal{H}}$ is the period domain of Enriques surfaces. The quotient $(\mathcal{D}(L_-) \setminus \tilde{\mathcal{H}})/O(L_-)$ (resp. $(\mathcal{D}(L_-) \setminus \tilde{\mathcal{H}})/\tilde{O}(L_-)$) is the moduli space of Enriques surfaces (resp. the moduli space of marked Enriques surfaces).

The case of Hessian quartic surfaces is similar. First define

$$(5.2) \quad \mathcal{D}(M) = \{[\omega] \in \mathbf{P}(M \otimes \mathbf{C}) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}$$

which is a disjoint union of two copies of the 4-dimensional bounded symmetric domain of type IV. We can consider $\mathcal{D}(M)$ as a subdomain of $\mathcal{D}(L_-)$ under the embedding $M \subset L_-$. We denote by Γ_M the orthogonal group $O(M)$ of M and by $\tilde{\Gamma}_M$ the kernel of the map

$$O(M) \rightarrow O(q_M).$$

Then $\Gamma_M/\tilde{\Gamma}_M \cong O(q_M) \cong \mathfrak{S}_5 \times \{\pm 1\}$. The quotient $\mathcal{D}(M)/\tilde{\Gamma}_M$ is the moduli space of *marked* Hessian quartic surfaces.

For a vector $r \in M^*$ with $r^2 < 0$, we also define

$$r^\perp = \{[\omega] \in \mathcal{D}(M) \mid \langle \omega, r \rangle = 0\}.$$

Let $a \in A_M$ and let m be a negative rational number. Define

$$\mathcal{H}_{a,m} = \sum_r r^\perp$$

where r moves over the set of all vectors in M^* satisfying $r \bmod M = a$ and $r^2 = m$. We call $\mathcal{H}_{a,m}$ the *Heegner divisor* of type a and m .

5.1. Proposition. *A generic point of the Heegner divisor $\mathcal{H}_{a,m}$ corresponds to the period of the Hessian quartic surfaces of the following cubic surfaces S . If $a = 0$ and $m = -2$, then S has a node. If $q_M(a) = 1$ and $m = -1$, then S has an Eckardt point. If $q_M(a) = m = -1/3$, then S has no Sylvester forms.*

Proof. The assertion follows from Dardanelli, van Geemen [12], Lemmas 2.2, 3.1, 5.1. Also we have seen the assertion for $q_M(a) = 1$ in Lemma 4.2. \square

It follows from Sterk [24], Corollary 3.3 that any two vectors r, s in M satisfying

$$r^2 = s^2, \langle r, M \rangle = \langle s, M \rangle =: p\mathbf{Z}, \quad (p > 0), \quad r/p \bmod M = s/p \bmod M$$

are equivalent under the action of $\tilde{\Gamma}_M$. In particular the image of each Heegner divisor

$$\mathcal{H}_{0,-2}, \mathcal{H}_{a,-1} \quad (a \in A_M, q_M(a) = 1), \mathcal{H}_{a,-1/3} \quad (a \in A_M, q_M(a) = -1/3)$$

in $\mathcal{D}(M)/\tilde{\Gamma}_M$ is irreducible. We denote by $\mathcal{D}(M)^\circ$ the complement of the Heegner divisors $\mathcal{H}_{0,-2}$ and $\mathcal{H}_{a,-1/3}$, $a \in A_M, q_M(a) = -1/3$ in $\mathcal{D}(M)$. Let

$$(5.3) \quad \Lambda = \{\lambda \in \mathbf{P}^4 \mid \Delta_{\text{sing}}(\lambda) \neq 0, \quad \lambda_i \neq 0, \quad i = 1, \dots, 5\}.$$

The symmetric group \mathfrak{S}_5 of degree 5 acts on Λ as permutations of the coordinate of \mathbf{P}^4 , and on $\mathcal{D}(M)^\circ$ as the action of $\Gamma_M/\tilde{\Gamma}_M \cong \mathfrak{S}_5 \times \{\pm 1\}$. Then the global Torelli type theorem for $K3$ surfaces and Proposition 5.1 imply the following proposition.

5.2. Proposition. ([17], Theorem 2.1) *The period map gives an \mathfrak{S}_5 -equivariant embedding from Λ to $\mathcal{D}(M)^\circ/\tilde{\Gamma}_M$.*

6. AUTOMORPHIC FORMS

In [19], the author constructed automorphic forms of weight 4 on $\mathcal{D}(L_-)$ with respect to the group $\tilde{\mathcal{O}}(L_-)$ by using the theory of automorphic forms due to Borchers [6]. We recall this briefly.

First we recall that A_{L_-} is isomorphic to the orthogonal direct sum of 5 copies of u . A 5-dimensional subspace V of A_{L_-} is called a *maximal totally singular subspace* if V is generated by mutually orthogonal non-isotropic vectors a_1, \dots, a_5 . By using Borchers theory [6], for each V , we can associate a holomorphic automorphic form F_V on $\mathcal{D}(L_-)$ of weight 4 with respect to $\tilde{\mathcal{O}}(L_-)$. Moreover the zero divisor of F_V is

$$(6.1) \quad \sum_{a \in V, q_{L_-}(a)=1} \tilde{\mathcal{H}}_a.$$

The linear system of these automorphic forms together with an another automorphic form of the same weight define an $\mathcal{O}^+(10, \mathbf{F})$ -equivariant morphism from $\mathcal{D}(L_-)/\tilde{\mathcal{O}}(L_-)$ to \mathbf{P}^{186} which is birational onto its image (In [19], there was a mistake pointed out and corrected by Freitag and Manni [15], Theorem 11.2). There are $3^3 \cdot 5 \cdot 17 \cdot 31$ maximal totally singular subspaces in A_{L_-} .

In the following, we consider the *restriction* of automorphic forms F_V to $\mathcal{D}(M)$. Recall that M is the orthogonal complement of $R \cong E_6(2)$ in L_- (Lemma 3.1). Denote by q_2 the restriction of a discriminant quadratic form q to the 2-Sylow subgroup. Then

$$(q_R)_2 \cong u \oplus u \oplus v, \quad (q_M)_2 \cong u \oplus v.$$

By using the relation $u \oplus u \cong v \oplus v$, we can see that

$$q_{L_-} \cong (q_R)_2 \oplus (q_M)_2.$$

Let a_1, a_2, a_3 be mutually orthogonal vectors with $q_R(a_i) = 1$ ($i = 1, 2, 3$) in A_R . Then a_1, a_2, a_3 are mutually orthogonal non-isotropic vectors in A_{L_-} . Let b_1, b_2 be mutually orthogonal vectors with $q_M(b_i) = 1$ ($i = 1, 2$) in M^*/M . Then

$$a_1, a_2, a_3, b_1, b_2$$

are mutually orthogonal non-isotropic vectors in A_{L_-} , and hence they generate a maximal totally singular subspace V in A_{L_-} . There are 15 pairs $\{b_1, b_2\}$ of mutually orthogonal non-isotropic vectors in M^*/M . Thus, for fixed a_1, a_2, a_3 , we have 15 maximal totally singular subspaces in A_{L_-} of this type.

Now we study the restriction of Heegner divisors appeared in (6.1). Let V be a maximal totally singular subspace generated by a_1, a_2, a_3, b_1, b_2 . Then V contains 16 non-isotropic vectors

$$\begin{aligned} & a_1, a_2, a_3, a_1 + a_2 + a_3, b_1, b_2, a_i + a_j + b_k \ (1 \leq i < j \leq 3, k = 1, 2), \\ & a_i + b_1 + b_2 \ (i = 1, 2, 3), a_1 + a_2 + a_3 + b_1 + b_2. \end{aligned}$$

Obviously $\tilde{\mathcal{H}}_a$ ($a = a_1, a_2, a_3, a_1 + a_2 + a_3$) vanishes along $\mathcal{D}(M)$. On the other hand, if r is a (-4) -vector in L_- with $r/2 \bmod L_- = b_j$, then $r \in M$ or the projection of r into M^* has a non-negative norm because the maximal norm of non-zero vectors in R is -4 . In the later case, the hyperplane r^\perp does not meet with $\mathcal{D}(M)$. Therefore $\tilde{\mathcal{H}}_{b_j}$ ($j = 1, 2$) cuts the Heegner divisor $\mathcal{H}_{b_j, -1}$ on $\mathcal{D}(M)$. In case $a = a_i + a_j + b_k$, $a_i + a_j$ is represented by $r/2$ with $r \in R$, $r^2 = -8m$, m a positive integer, because $a_i + a_j$ is non-zero isotropic vector and the maximal norm of non-zero vectors in R is -4 . This implies that b_k is represented by a positive norm vector in M , and hence $\tilde{\mathcal{H}}_a$ does not intersect with $\mathcal{D}(M)$ and its boundary. Similary in case $a = a_i + b_1 + b_2$ or $a = a_1 + a_2 + a_3 + b_1 + b_2$, a_i and $a_1 + a_2 + a_3$ are represented by $r/2$ with $r \in R$, $r^2 = -4m$ where m is a positive integer. This implies that $b_1 + b_2$ is represented by a non-zero isotropic vector or a positive norm vector in M , and hence $\tilde{\mathcal{H}}_a$ does not meet with the interior of $\mathcal{D}(M)$.

Now recall that E_6 contains 72 roots ([8], Planche V), and hence $R = E_6(2)$ contains 72 (-4) -vectors. On the other hand, the number of non-isotropic vectors of the quadratic form $(q_R)_2 = u_2 \oplus u_2 \oplus v_2$ is 36. By sending each (-4) -vector $\pm r$ in R to $r/2 \bmod R$ in $(q_R)_2$, we have a bijective correspondence between the set of (-4) -vectors in R modulo ± 1 and the set of non-isotropic vectors in $(q_R)_2$. Thus each F_V vanishes along 4 hyperplanes $(\pm r)^\perp$ where $r \in R$ with $r^2 = -4$ corresponding to 4 non-isotropic vectors $a_1, a_2, a_3, a_1 + a_2 + a_3$. By a method given in [7], that is, by first dividing F_V by a product of linear forms vanishing on the divisors associated to the four (-4) -vectors and restricting it to $\mathcal{D}(M)$ we can get an automorphic form $F_V|_M$ on $\mathcal{D}(M)$ with respect to the group $\tilde{\Gamma}_M$. Then the weight of $F_V|_M$ is the weight of F_V plus half the number of (-4) -vectors in R corresponding to $a_1, a_2, a_3, a_1 + a_2 + a_3$, that is, $4 + 4 = 8$. We now conclude

6.1. Theorem. *Let V be a maximal totally singular subspace generated by $\{a_1, a_2, a_3, b_1, b_2\}$. Then $F_V|_M$ is a holomorphic automorphic form on $\mathcal{D}(M)$ of weight 8 with respect to the group $\tilde{\Gamma}_M$ whose zero divisor is $\mathcal{H}_{b_1, -1} + \mathcal{H}_{b_2, -1}$.*

As mentioned in Proposition 5.1, a generic point of the Heegner divisor $\mathcal{H}_{b_i, -1}$ is the period of the Hessian quartic surface of a cubic surface with an Eckardt point. Assume that non-isotropic vector b_1 or b_2 corresponds to the condition $\lambda_i = \lambda_j$ or $\lambda_k = \lambda_l$ respectively. Then the condition that b_1 is

orthogonal to b_2 is equivalent to that all i, j, k, l are different (see Lemma 4.2). By identifying Λ and an open subset in $\mathcal{D}(M)$ (see Proposition 5.2), we have the following theorem.

6.2. Theorem. *As divisors on Λ ,*

$$(F_V|_M) = ((\lambda_i - \lambda_j)(\lambda_k - \lambda_l)).$$

6.3. Remark. *We can easily see that fifteen $(\lambda_i - \lambda_j)(\lambda_k - \lambda_l)$, where i, j, k, l are distinct, generate a 5-dimensional space W of quadrics on \mathbf{P}^4 whose base locus is the sum of five lines defined by $\lambda_i = \lambda_j = \lambda_k$ ($1 \leq i < j < k \leq 5$) meeting at $(1 : 1 : 1 : 1 : 1)$. Let*

$$\xi = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_5), \eta = (\lambda_1 - \lambda_3)(\lambda_4 - \lambda_5), \zeta = (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_5),$$

$$\xi' = (\lambda_1 - \lambda_2)(\lambda_4 - \lambda_5), \eta' = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_5), \zeta' = (\lambda_1 - \lambda_4)(\lambda_3 - \lambda_5).$$

The $\xi, \xi', \eta, \eta', \zeta, \zeta'$ generate W and satisfy the relations

$$\xi + \eta + \zeta = \xi' + \eta' + \zeta', \quad \xi\eta\zeta = \xi'\eta'\zeta'.$$

These relations define the Segre cubic 3-fold S_3 (Baker [3], Hunt [16], §3.2.2). Since the restriction of W on a general hyperplane \mathbf{P}^3 in \mathbf{P}^4 is the linear system of quadrics with five points as its base locus, it gives a birational map from \mathbf{P}^3 to S_3 (Hunt [16], Theorem 3.2.1). Thus the linear system W gives a dominant rational map from \mathbf{P}^4 to S_3 .

6.4. Remark. *Borcherds [5] constructed an automorphic form Φ_4 on $\mathcal{D}(L_-)$ of weight 4 whose zero divisor is the Heegner divisor $\tilde{\mathcal{H}}$ associated to (-2) -vectors in L_- . Since $R = E_6(2)$ has no (-2) -vectors, the restriction of Φ_4 defines an automorphic form on $\mathcal{D}(M)$ of weight 4. Let r be a (-2) -vector in L_- and let*

$$r = r_1 + r_2, \quad r_1 \in R^*, \quad r_2 \in M^*.$$

Assume that $r_1 \neq 0$. Since R contains no (-2) -vectors, $r_2 \neq 0$. Since $M \oplus R$ has index 3 in L_- , $3r_1 \in R$, $3r_2 \in M$. By Lemma 4.1, $q_M(r_2 \bmod M) = -4/3$. Hence $q_R(r_1 \bmod R) = -2/3$. Since R contains no (-6) -vectors, $(r_1)^2 \leq -8/3$ and hence $(r_2)^2 > 0$. Therefore r^\perp does not intersect with $\mathcal{D}(M)$. This implies that if the projection of a (-2) -vector in L_- into M^ has a negative norm, then it is a (-2) -vector. Thus the restriction of Φ_4 is an automorphic form on $\mathcal{D}(M)$ of weight 4 whose zero divisor is the Heegner divisor $\mathcal{H}_{0,-2}$ associated to (-2) -vectors in M . The corresponding cubic surfaces are nodal (Proposition 5.1).*

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